COMBINATORIAL GAME THEORY

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ABSTRACT. This document reviews CGT, and provides proofs for the fundamental results. It follows the structure of section 3.1 of "Computer Go as a Sum of Local Games: An Application of Combinatorial Game Theory" (1995), by Martin Müller `http://www.cs.ualberta.ca/~mmueller/ps/th11006.pdf.zip`, but some definitions are filled in from other sources, and the proofs are my own.

When a game of perfect information is played alone by omniscient players, it can be summarised by its result. Chequers is a draw, for example. However, the result alone is a completely inadequate summary when the game is played as part of a sum.

A sum of games is itself a game, played between two players, who play alternately. On each move they may make a move in exactly one of the games in the sum. The winner, as in chequers, is the last person with a legal move. Sum games are interesting because many games (but not chequers) naturally decompose into a sum game. Examples include Nim, Sprouts, Dots and Boxes, Domineering, Clobbers, and most interestingly a variant of Go.

Viewing one game in the sum in isolation, the players do not typically move alternately. The move order depends on what is going on in the other games in the sum. Thus the result under alternating play is not enough. Games can nonetheless be summarised; most of the details of the gameplay are unnecessary to compute the result of the sum. CGT is the study of these summarised games.

1. BASIC DEFINITIONS AND NOTATION

Definition 1.1. Game

A *game* is played between two players called Left and Right. It is defined by its left and right *options*, *i.e.* the positions to which Left and Right can move:

$$G = \{L_1, L_2, \dots \mid R_1, R_2, \dots\}$$

This definition is recursive; L_i and R_j are themselves games.

The set $\{L_1, L_2, \dots\}$ of left options of G is written G_L, and similarly the set of right options is written G_R. Note that these are sets of games and not games.

Definition 1.2. Inverse of a game

The *inverse* of a game is the game with the players reversed:

$$-G = \{-R \text{ for } R \in G_R \mid -L \text{ for } L \in G_L\}$$

Note that $- - G = G$.

It is conventional to define names for some common games.

Definition 1.3. Integers

The game $\{|\}$ is called 0. Note that $-0 = 0$. For each non-negative integer n, the game $n + 1$ is defined in terms of the game n by $n + 1 = \{n\ |\}$, and the game $-n$ is defined by negating n.

There are games corresponding to some fractional numbers too, such as $\frac{1}{2} = \{1 \mid 0\}$ and $\frac{1}{4} = \left\{\frac{1}{2} \mid 0\right\}$.

Definition 1.4. Sum of games

The *sum* $G + H$ of games G and H is defined by:

$G{+}H = \{L + H \text{ for } L \in G_L, G + L \text{ for } L \in H_L \mid R + H \text{ for } R \in G_R, G + R \text{ for } R \in H_R\}$

The *difference* $G - H$ is simply $G + (-H)$.

A move in the sum game is therefore a move either in G or in H.

Note that addition of games is commutative and associative, that adding 0 to a game does not change it, and that negation distributes over addition.

Lemma 1.5. *Sum of integers*

Let m and n be non-negative integers. Using the notation (x) to mean the game corresponding to the integer x, the sum of the games (m) and (n) is the game $(m + n)$.

The cases $m = 0$ and $n = 0$ are easy.

The case $m = 1$ can be proved by induction on n. For the inductive step, suppose $(1) + (n) = (n + 1)$. Then the sum of $(1) = \{(0) \mid\}$ and $(n + 1) = \{(n) \mid\}$ is

$$\begin{aligned}
&\{(0) + (n+1), (1) + (n) \mid\} \\
=&\{(n+1), (n+1) \mid\} \\
=&\{(n+1) \mid\} \\
=&(n+2)
\end{aligned}$$

Cases $m > 1$ can then be proved by induction on m.

Note, however, that $(1) - (1) = \{-1 \mid 1\} \neq \{\mid\} = (0)$. In general, addition of games corresponding to numbers with opposite sign does not give another number. This will be fixed in section 3.

2. Comparing games

Two games G and H can be compared according to whether left would prefer G to H in any sum. This is a partial order; there exist incomparable pairs of games in which one game would be better in some sums and the other in other sums.

Left "prefers" a game if he is more likely to win under alternating play.

Definition 2.1. Winning

The predicates leftwins(G) and rightwins(G) are defined mutually recursively on the game G:

$$\begin{aligned}
\text{leftwins}(G) &\iff \exists L \in G_L. \neg\text{rightwins}(L) \\
\text{rightwins}(G) &\iff \exists R \in G_R. \neg\text{leftwins}(R)
\end{aligned}$$

Note that this definition ensures that a player loses when they run out of moves.

Note that leftwins(G) and rightwins(G) are not opposites; the former determines who wins when Left starts, but Right starts for the latter. In particular, neither leftwins(0) nor rightwins(0), yet both leftwins($\{0 \mid 0\}$) and rightwins($\{0 \mid 0\}$).

Lemma 2.2. *Self-difference*

For any game G:

$$\neg \textit{leftwins}(G - G)$$

Proof. Induction on the structure of G.

Suppose, hoping for a contradiction, that G is the smallest game such that leftwins$(G-G)$. Then there is a winning move to some game $L \in (G-G)_L$ such that $\neg$rightwins(L).

There are two cases to consider: either $L = G' - G$ for some $G' \in G_L$ or $L = G - G'$ for some $G' \in G_R$. These cases are symmetrical (under negation of G) so without loss of generality let's consider only the former.

Let $R = G' - G'$. Certainly $R \in (G' - G)_R$, *i.e.* Right can move from L to R. However, $\neg$rightwins(L) so this must be a losing move, *i.e.* leftwins(R). However, G' is smaller than G, contradicting our initial assumption. □

Definition 2.3. Dominates

Say a game G *dominates* a game H, and write $G \geq H$, iff for all games X:

$$\begin{aligned} \text{leftwins}(H+X) &\implies \text{leftwins}(G+X) \\ \text{rightwins}(G+X) &\implies \text{rightwins}(H+X) \end{aligned}$$

Note that $\geq$ is reflexive and transitive, that it is a pre-congruence under $+$, and that $G \geq H$ iff $-H \geq -G$.

The definition of $\geq$ is awkward because it involves a quantification over all games. This burden of proof is carried by the following theorem:

Theorem 2.4. *Domination is inductive*

$G \geq H$ iff $\neg$leftwins$(H - G)$.

Proof. If $G \geq H$ then by putting $X = -G$ in definition 2.3 we obtain leftwins$(H - G) \implies$ leftwins$(G - G)$ which is false by lemma 2.2. This completes half the proof.

For the other half of the proof, suppose that $\neg$leftwins$(H - G)$ and, hoping for a contradiction, that X is a game that breaks at least one of the implications in definition 2.3. We will proceed by induction on $H - G$ and on X; therefore further suppose that $H - G$ is as small as possible, and that X is as small as possible given $H - G$.

There are two cases to consider. Either leftwins$(H + X)$ and $\neg$leftwins$(G + X)$, or rightwins$(G + X)$ and $\neg$rightwins$(H + X)$. These cases are symmetrical (under interchange of G and $-H$) so let's consider only the former.

From leftwins$(H + X)$ we can find a winning move from $H + X$ to some game $L \in (H + X)_L$ such that $\neg$rightwins(L). The winning move is either in H or in X.

We can dismiss the possibility that $L = H + X'$ for some game $X' \in X_L$, because X' is smaller than X. We know $\neg$leftwins$(H - G)$ and $\neg$rightwins$(H + X')$ so we cannot also have rightwins$(G + X')$. From $\neg$rightwins$(G + X')$ we deduce leftwins$(G + X)$, contradicting our hypothesis.

The remaining possibility is that $L = H' + X$ for some game $H' \in H_L$. Then this move in $H + X$ is also a move in $H - G$, specifically to $H' - G$. We assumed $\neg$leftwins$(H - G)$, so it must be a losing move, *i.e.* rightwins$(H' - G)$. We can therefore find a winning move from $H' - G$ to some game $R \in (H' - G)_R$ such that $\neg$leftwins(R). This winning move is either in H' or in G.

We can dismiss the possibility that $R = H' - G'$ for some $G' \in G_L$ because $H' - G'$ is smaller than $H - G$. We know $\neg$leftwins$(H' - G')$ and $\neg$rightwins$(H' + X)$ so we cannot also have rightwins$(G' + X)$. From $\neg$rightwins$(G' + X)$ we deduce leftwins$(G + X)$, contradicting our hypothesis.

We can also dismiss the remaining possibility that $R = H'' - G$ for some $H'' \in H'_R$ because $H'' - G$ is smaller than $H - G$. We know ¬leftwins($H'' - G$) and ¬leftwins($G + X$) so we cannot also have leftwins($H'' + X$). From ¬leftwins($H'' + X$) we could deduce rightwins($H' + X$), contradicting the definitive property of H'. □

3. EQUIVALENCE

If two games dominate each other, they are interchangeable in any context. We may therefore regard them as equal.

Definition 3.1. Equality

Say two games G and H are equivalent, and write $G \simeq H$, iff $G \geq H$ and $H \geq G$.

Games become much nicer objects if we quotient out the equivalence relation $\simeq$, *i.e.* if we treat games as equal whenever they are merely equivalent.

Theorem 3.2. *Abelian group*

Games form an abelian group up to $\simeq$, with $+$ as the composition operator, negation as the inverse operator, and 0 as the identity element.

Proof. We need to prove the following propositions for all games F, G and H:

$$F + (G + H) \simeq (F + G) + H$$
$$G + H \simeq H + G$$
$$G + 0 \simeq G$$
$$G - G \simeq 0$$

All follow easily from the definitions except the last, which follows from lemma 2.2 and theorem 2.4. □

Note that although $1 - 1 \neq 0$ (in game arithmetic), we do instead have $1 - 1 \simeq 0$. In fact, lemma 1.5 and theorem 3.2 together prove that addition and subtraction work up to $\simeq$ for all integer games exactly as they do for integers. This fixes the strange behaviour of numbers observed in section 1.

4. CANONICAL FORM

To be continued...